Crab Canons: The Backwards Walking Crustacean with a Twist

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Acknowledgements

This senior capstone experience is a true reflection of the dedication and passion that I have cultivated during my time at Washington College. Whenever I discussed my majors with anyone outside of the mathematics and music departments, so many of them met me with such curiosity—or in some cases confusion—as to how these two subjects could even collide in a way that wasn't counting the beats of a measure of music. It is my pride and honor to present the following thesis as a representation of just one of the many intersections of math and music. But this thesis could not have been possible without the incredible support system that has carried me throughout this entire process, and I wish to thank each of the people who have contributed to the success of my capstone.

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much more fun. Without Dr. Wilson's excitement and guidance, readers would not get to enjoy the wonderful canon that you will see in a later section of this paper.

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Table of Contents

I. Introduction

When initially researching mathematical happenings within a musical context, I was pointed toward a YouTube video of Johann Sebastian Bach's "Canon 1 a 2" from *The Musical Offering* (1747). The video begins by playing the original melody line of the canon and then introduces a retrograde of the line by playing the melody backwards, hence the canon's nickname of the "crab canon." However, in the transformation into a Möbius strip, the retrograded line is then also inverted and placed on the backside of the strip before twisting and linking the ends together. In topology, one could argue that a true Möbius strip was not created if a split in the strip was utilized before the twist, but scholars highly regard Bach's Crab Canon as a true Möbius strip.

My thesis explores this topological phenomenon of Bach's Crab Canon and whether it is truly a composition on a Möbius strip. First, we establish all of the necessary musical terms for understanding Bach's composition. These terms are used as preliminaries to guide our perception of aesthetics in musical composition and for further analysis in later sections. From here, we define a Möbius strip and begin deconstructing both sides of the argument for whether Bach's Crab Canon can be transposed onto one. After analyzing the use of mathematical transformations in the construction of a Möbius strip, we further explore the significance of symmetries in music through these transformations. Through mathematical concepts such as surfaces and transformations, music visualization can be expanded to aid audiences in understanding the compositional choices of composers. Finally, we gather all of these concepts to analyze a brief composition that I made to demonstrate musical transformations and surfaces.

II. Musical Preliminaries

i. A Foreword

There are many ways in which we can perceive music, and how we interpret "correctness" in sound is subjective to the listener and their musical training/knowledge. As a Western-trained musician, and for the purposes of this paper, all musical preliminaries and their use in later sections will be defined based on their use and definitions in Western tonal music.

ii. Intervals

An interval is defined as the distance in pitch between two notes. This distance is determined by (1) the number of diatonic scale degrees included and (2) the number of semitones or half-steps between the pitches. A diatonic scale is built upon seven pitches that are adjacent to one another on the circle of fifths, where each letter name represents a single pitch, and the proceeding scale is built upon distinct patterns of whole tones and semitones to create a scale. Regardless of the key signature, the diatonic scale assigns a degree name to each pitch: tonic, supertonic, mediant, subdominant, dominant, submediant, leading tone, and returning to tonic. Once the diatonic scale is determined, we can build basic intervals from the pitches as shown in Figure 2.1.1.

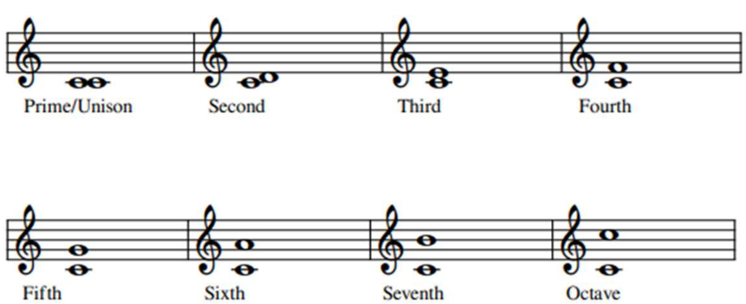


Figure 2.1.1 There are eight main intervals based on the diatonic scale degrees. It is possible to go beyond an eighth, (i.e., a ninth, tenth, etc.), but we consider these to be compound intervals since they are an octave plus a smaller interval. For example, a tenth is built from an octave plus a second.

To further define an interval, we count the number of semitones between the two pitches to add a qualifying adjective: perfect, major, minor, diminished, or augmented. To return to the octave, Western music follows a system of twelve semitones.

	Diminished	*Minor*	*Perfect*	*Major*	*Augmented*
Second	0	1		2	3
Third	2	3		4	5
Fourth	4		5		6
Fifth	6		7		8
Sixth	7	8		9	10
Seventh	9	10		11	12
Octave	11		12		13

Table 2.1.2. The number of semitones needed to create each type of interval.

If interval types have the same number of semitones but different names, they are considered enharmonically equivalent. Thus, a pair of intervals may be enharmonically equivalent if the specific pitches that make up the first interval are enharmonically equivalent to the pitches of the second interval e.g., the interval a♭ – b♭, a major second, is enharmonically the same as the interval g♯ – b♭, a diminished third, as a♭ and g♯ are enharmonically equivalent pitches.

iii. Consonance and Dissonance

When two or more pitches are stacked together, they create a chord. The perceived stability or instability of the chord is then dependent on the interval they are built upon. This perception of stability is identified as consonance or dissonance. Consonant intervals are those that provide aural stability while dissonant intervals provide an impression of tension or clash. Primarily, an interval will either be consonant or dissonant, and a consonant interval can then either be perfect or imperfect. The consonant intervals include the unison, third, fifth, sixth, and octave. Of these, the unison, fifth, and octave are considered perfect consonances due to their harmonic relationship found in the natural overtone series. The sixth and third intervals are thus imperfect. The remaining intervals of the second, fourth[1], diminished fifth (also equivalent to an augmented fourth or a tritone), and seventh are all considered dissonant intervals.

iv. Types of Motion

To create this consonant motion, we will define the types of motion that were developed for their use in counterpoint. The three types of motion defined by Johann Fux in his 1725 published study of counterpoint, *Gradus ad Parnassum*, are direct motion, contrary motion, and oblique motion:

1. *Direct motion* occurs when two or more parts ascend or descend in the same direction by step or skip.
2. *Contrary motion* occurs when one part ascends by step or skips while the other descends—or vice versa.
3. *Oblique motion* occurs when one part moves by step or skips while the other remains stationary on the same note (1971, 21-22).

Though we begin with describing direct motion as any form of ascending or descending motion in the same direction, over time music theory specified two separate motions to describe when two or more parts move in the same intervallic distance versus when they move in different intervals. Thus, we further define similar motion as two or more parts that ascend or descend in

[1] Unless otherwise indicated, a fourth usually refers to a perfect fourth, made of five semitones. In Western music, a fourth can be considered a perfect interval and a dissonance at the same time (Drabkin 2001). The interval is not so much dissonant as "unstable," but due to its harmonic relationships with the third and fifth and the harmonic and arithmetical divisions that Fux defines in *Gradus ad Parnassum*, the fourth creates a need to resolve in order to create consonance.

the same direction in different intervals; and parallel motion as two or more parts that ascend or descend in the same direction in the same constant interval.

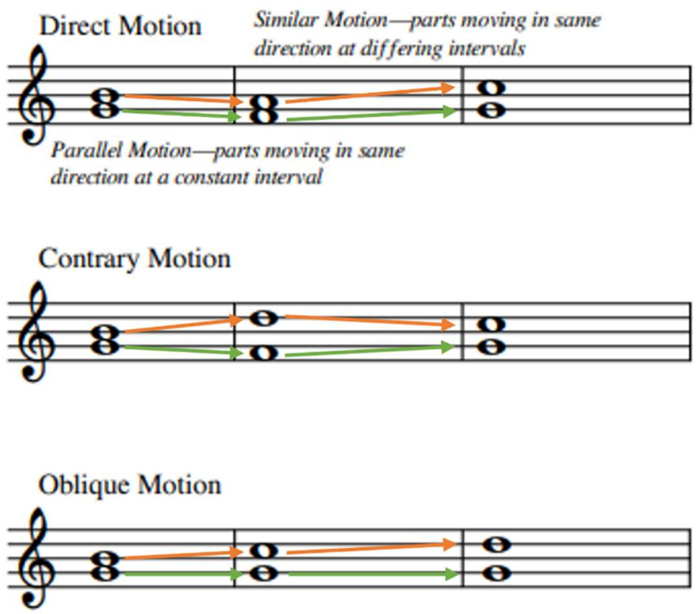


Figure 2.4.1. The three primary forms of motion with the expanded forms of direct motion.

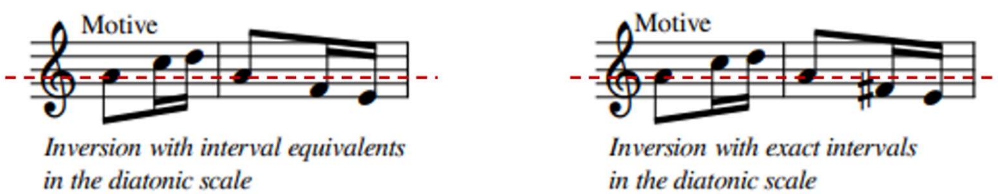
v. Inversions and Retrogrades

Before we can discuss mathematical transformations found in music, we will first define an inversion and a retrograde as they are found in music.

A complement or an *inversion* of an interval occurs when the sum of the original interval and the inverted interval form a third, fixed interval with respect to which the inversion takes place. Unless otherwise specified, an inversion is measured with respect to the octave. Thus, an interval is inverted by raising the lower pitch class above the upper one, specifically by an octave. Since we know an octave contains 12 semitones, the sum of two intervals related by inversion is 12. For example, the inversion of a major third would be a minor sixth, as Table 2.1.2 states a major third contains 4 semitones and a minor sixth contains 8 semitones, totaling 12 semitones. With respect to the octave, the inversion of any interval of n semitones is the complement of n with a modulus of 12, or $12 - n$. Alternatively, we may think of the sum of the two interval numbers as always totaling 9. As a result, perfect intervals will always yield perfect intervals, major intervals will yield minor, and minor will yield major.

Intervals may also perform inversions with respect to an interval other than the octave. These inversions follow similar principles to inversions at the octave. For instance, in inversion at the twelfth, the lower pitch is raised a twelfth rather than an octave.

The inversion of a melody is one in which the contour of a melody is the mirror image of the original melody. We look at mirror images as reflections over a horizontal axis with respect to a line or space on the staff. Thus, if a melody rises, its inversion falls, and vice versa. The intervals between successive pitches may stay exact, similar to how intervals must stay constant in parallel motion, though it is especially common in tonal music to find a motion that follows the contour of the inverted interval but is not exactly the same intervallic distance from the horizontal axis of reflection.



Example 2.5.1. Two inversions of the same motive, reflecting over the horizontal axis with respect to the pitch A4. The original motive moves from *a – c – d*, with a minor third between *a – c* and a major second between *c – d*. Inversion 1 (left) shows an inversion with interval equivalents, but not exact intervals: *a – f – e* has a major third between *a – f* and a minor second between *f – e*. The use of interval equivalents is more often found in tonal music. Inversion 2 (right) shows an inversion with exact intervals as the original motive: *a – f# – e* has a minor third between *a – f#* and a major second between *f# – e*.

(See Figure 4.2.1 in section 4.2 for more on inversions and retrogrades as mathematical transformations in music.)

In simplest terms, a *retrograde* is the backwards or reverse of a melody. A retrograde begins with the last note and ends with the first. In Latin, a retrograde is known as *cancrizans* or "crab motion" for its similarity to how a crab walks backwards.

Additionally, we can perform a *retrograde inversion* in which a melody is reflected over a horizontal axis (with respect to any interval of choice, but most commonly over the middle line of the staff) and then presented backwards from the last to the first note. These three forms of musical transformations are discussed further in section 4.

vi. Canons

Musically, there are many different definitions of a canon, though for this paper, a *canon* refers to the imitation of a complete subject by one or more voices at fixed intervals of pitch and time. The leading voice, known as the *dux*, carries the original melody that is then imitated by the successive voice, known as the *comes*. As defined in "The Harvard Concise Dictionary of Music and Musicians," (Randel 1999, 110-111), there are five elements that classify canons:

1. The time between entries—whether a canon enters at the half note, the whole note, etc.
2. The interval between entries—whether a canon enters at the unison, at the fifth above, etc.
3. Transformations of the subject, as follows:
    a. Inversion;
    b. Retrograde;
    c. Retrograde inversion;

d. Augmentation, where the note values of the *comes* are longer than the *dux* by a fixed ratio;
e. Diminution, where the note values of the *comes* are shorter than the *dux*; and
f. Mensuration, where the *dux* is interpreted simultaneously in different proportions with a possible result that the temporal relationship between voices may shift due to the differing interpretation of individual note values.

4. Ending—finite canons have a definite ending that may either add notes to the *dux* to make up for the time lag created by the succeeding voices or may add a short coda; infinite canons (such as the perpetual canon, circle canon, round, etc.) return to the beginning with an arbitrary ending often shown by a fermata. While finite and infinite canon endings are relative to time in the horizontal plane, other instances of endings are relative to the vertical plane. Some canons may modulate and end in a different key than they originally started in, as in the case of canons like the spiral canon.
5. Number of canons—canons can be combined with other canons to create group or compound canons. Compound canons are indicated by the number of canons (a double canon has two, a triple canon has three, etc.) and by the number of parts. Thus, a two-part double canon has four parts and is considered a canon "four-in-two."

A common use of a canon is in children's nursery rhymes where the same melody line is layered on top of itself but with a set time interval in between each iteration. Nursery rhymes such as "Row, Row, Row Your Boat" and "Frère Jacques" are two well-known perpetual canons that show how cyclic a melody can be without a clear ending.

Example 2.6.1. Frère Jacques begins with a single voice and introduces a canon after two measures. Perpetual canons, like so, can often add as many voices as one would like, and because they lack a clear ending, these canons can theoretically go on infinitely.

vii. Crab Canons – The "Palindrome" of Music

While the name may sound like a funny device used to launch crustaceans, *crab canons* are in reference to crabs' ability to walk backwards, as mentioned in the definition of retrogrades. Crab canon compositions are a special type of musical canon that are conceptually similar to a palindrome for their ability to be read forward or backward. However, a true palindrome refers to a word or phrase that reads the same regardless of direction, such as "racecar" or "never odd or even." But the retrograde of a crab canon does not read the exact same as the prime, or else it would be that the prime equals the retrograde. So, we may use the term "semordnilap" to refer to the palindrome-like nature of a crab canon. *Semordnilap* ("palindromes" spelled backwards) is a name that was arguably coined around 1961 in Martin Gardner's revision of Charles Carroll Bombaugh's "Oddities and Curiosities of Words and Literature" to refer to

> A word, phrase, or sentence that has the property of forming another word, phrase, or sentence when its letters are reversed. A semordnilap differs from a palindrome

> in that the word or phrase resulting from the reversal is different from the original word or phrase (Wiktionary).

Examples of pairs of semordnilaps include “loop, pool” and “peels, sleep.” Since a retrograde of a melody creates a new melody rather than a repetition of the prime, we may think of the two voices in a crab canon as a pair of semordnilaps.

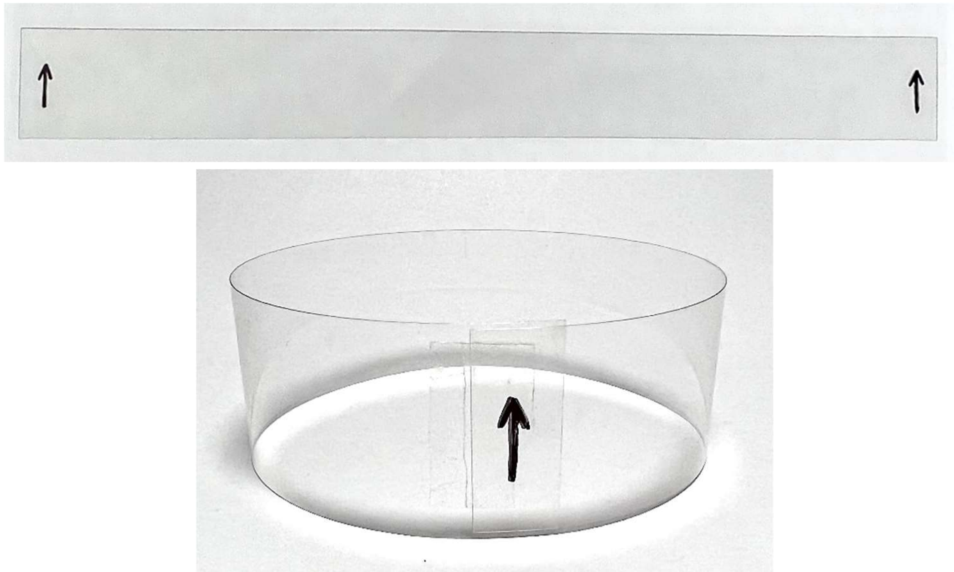
In this paper, we will be analyzing the canonical properties of Bach’s “Crab Canon” in the following section. This canon is considered a perpetual canon, such that the imitation of the melody recurs infinitely, circularly, and without end. In a perpetual crab canon, “the voices run through the same score in opposite directions, starting again from the beginning when they reach the end” (Cadeddu and Cannas 2017, 238).

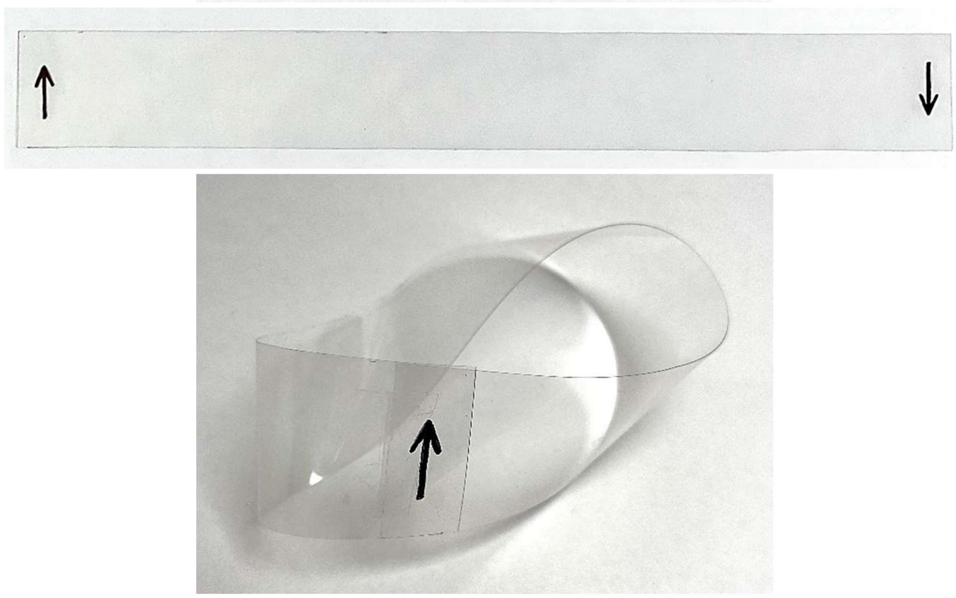
III. The Möbius Strip

i. What is a Möbius Strip?

In topology, there is a surface known as the Möbius[2] strip that contains one continuous side formed from the joining of two ends of a rectangular strip after twisting one end 180º. A strip can be twisted any number of times, but the outcomes vary uniquely depending on the number of twists. If we perform an even number of twists, the strip becomes two strips, with a “front” and a “back” side. If we perform an odd number of twists, the strip remains as one continuous side.

[2] Though the Möbius strip is named after mathematician August Ferdinand Möbius for his discovery of the strip in 1858, it is worth noting that mathematician Johann Benedict Listing also discovered the strip the same year but is not widely credited when referencing the strip. Listing’s additional contributions to mathematics include coining the term “topology” and the Listing numbers.





Figures 3.1.1 and 3.1.2. Figure 3.1.1 (top) depicts a strip being connected without any twists, creating a cylindrical surface. This figure clearly has a “front” and a “back” side. Figure 3.1.2 (bottom) depicts a strip being connected after one 180º twist is completed, which creates a Möbius strip. This figure contains one continuous side.

ii. Bach’s Crab Canon on a Möbius Strip

To understand how a musical composition could fit on the surface of a Möbius strip, we turn to a YouTube video of Bach’s “Canon 1 a 2” from *The Musical Offering* (1747). The video begins by playing the original melody line of the canon and then introduces a retrograde of the line by playing the melody backwards, hence the canon’s nickname of the “crab canon.”

However, in the transformation into a Möbius strip, the retrograded line is then also inverted and placed on the backside of the strip before twisting and linking the ends together. In the realm of topology, I was unsure of whether we could consider this a true Möbius strip since the strip was cut and rearranged before being twisted. It turns out that scholars are also torn between whether Bach's Crab Canon is a true Möbius strip. Whether the piece is or is not a Möbius strip, we will explore each argument and demonstrate how mathematical surfaces can be utilized in musical analysis.

Figure 3.2.1 A YouTube video uploaded in 2009 demonstrates Bach's Crab Canon being played on a Möbius strip with a silver and a gold marker following the two voices playing the twist. ("J.S. Bach - Crab Canon on a Möbius Strip")

iii. All Crab Canons Can be a Möbius Strip

As we have seen in section 2.7, the crab canon is built upon the simultaneous forward and backward layering of the same melodic line. The YouTube video representation of Bach's Crab Canon on a Möbius strip initially had me convinced that it worked. To construct a Möbius strip with any perpetual crab canon, we follow the steps detailed by Lucio Cadeddu and Sonia Cannas (2017, 238-39):

1. Let $ABCD$ be a rectangular strip of paper with $AB = DC > BC = AD$ where the whole score is on one musical stave containing both voices of the canon.
2. Divide the score in half by cutting the strip of paper between bar $n/2$ and $n/2 + 1$ with the first half of the score on rectangle $AM'N'D$ and the second half on rectangle $M''BCN''$.
3. Glue the two strips together by identifying and aligning $AM' \equiv CN''$ and $DN' \equiv BM''$.
4. From the new rectangle obtained, having the two halves of the canon on either side of the rectangle, we construct a Möbius strip by identifying A with N' and M' with D.

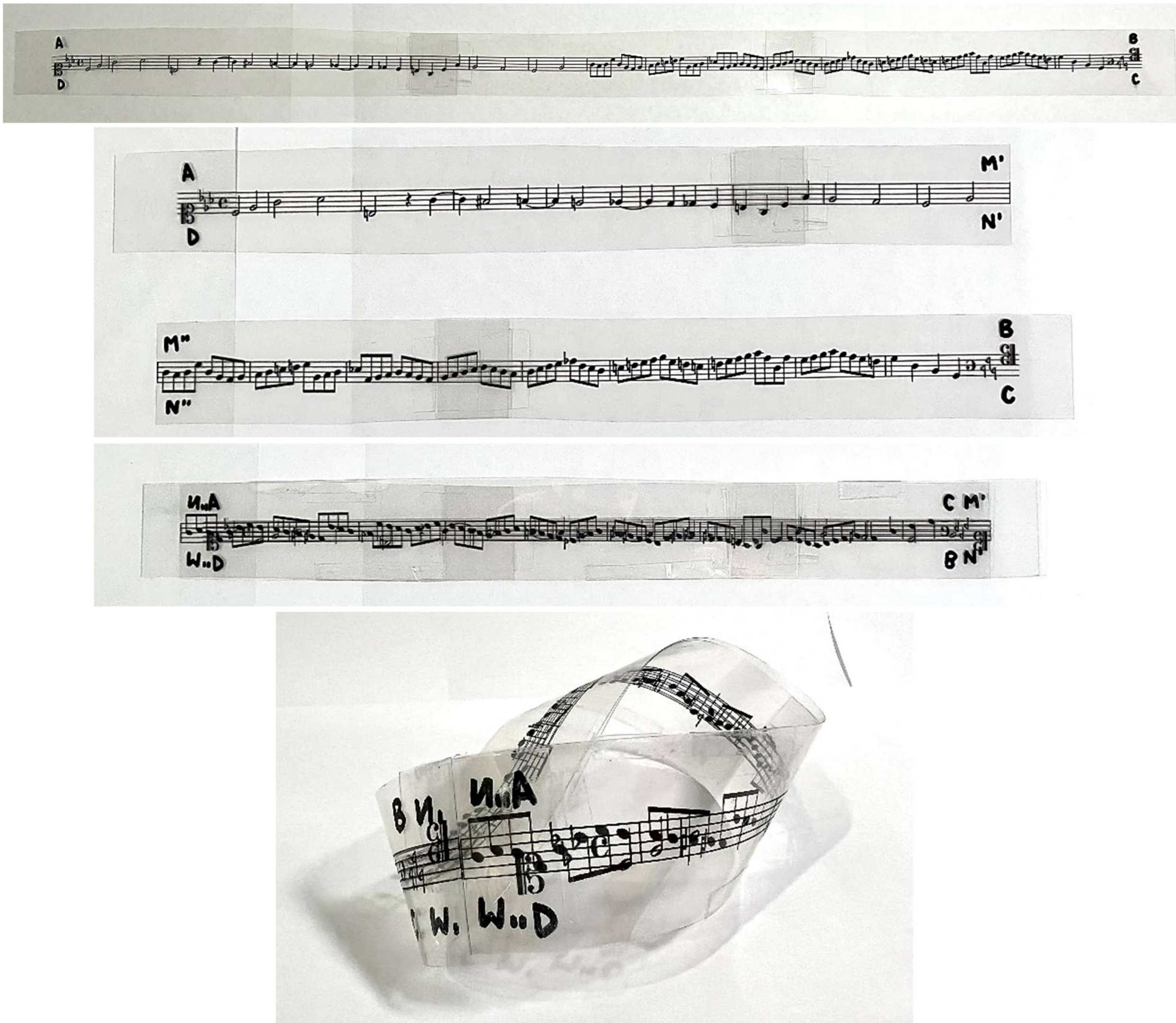


Figure 3.3.1. Using the four steps provided by Cadeddu and Cannas, Bach's Crab Canon is transformed into a Möbius strip. We begin with the piece as a single rectangle and then split it between measures 9 and 10, align and connect $AM' \equiv CN''$ and $DN' \equiv BM''$, and then twist the rectangle into a Möbius strip.

With this strip, we are able to play the canon both forward and backward, and it will create one continuous piece that loops infinitely. However, this method of creating the Möbius strip using Bach's Crab Canon only works because of step three, when we attach the second rectangle $M''BCN''$ to the first rectangle $AM'N'D$. By aligning $AM' \equiv CN''$ and $DN' \equiv BM''$, this allows for the original melodic line (which we define as the prime) to be read continuously without any musical transformations occurring. Notice that if we were to instead align $AM' \equiv BM''$ and $DN' \equiv CN''$, then at the midway point where the prime was split, the second half of the prime would be written upside down (Cadeddu and Cannas 2017, 239), which would sound like a retrograde inversion.

Figure 3.3.2. Aligning $AM' \equiv BM''$ and $DN' \equiv CN''$ results in a retrograde inversion, rather than just a retrograde.

To construct Bach's Crab Canon on a Möbius strip, we follow the instructions laid out above by splitting the score between measures 9 and 10 and attaching the two new rectangles together with the score facing outwards. The attaching must be done such that after reading the first nine measures, the second rectangle must be rotated 180º with respect to the long edge of the rectangle (Cadeddu and Cannas 2017, 240). Once connected, we are able to read both voices of the canon; going on the strip in one direction obtains the *dux*, while the reverse produces the *comes*. If reading in both directions, we then obtain the whole canon on a Möbius strip.

iv. A Cylinder in Disguise

The argument that Bach's Crab Canon is a Möbius strip seems convincing enough given Cadeddu and Cannas's clear-to-follow instructions, and I, too, saw no issue with the construction of the twist until coming across a column from the American Mathematical Society (Phillips 2016). The column was based on the work of Eric Altschuler and Anthony Phillips, who determined that the crab canon is actually a cylinder and not a Möbius strip (2015, 58).

In the steps to create a Möbius strip from the above section, we split the score into two rectangles that were glued together before the twist, but Altschuler and Phillips claim that this construction of a Möbius strip can be applied to any repeating piece of music so that the score can be read on one side and then the other. Topologically, once we can distinguish one side of the Möbius strip from the other, it becomes a connected double cover of the strip. The resulting cylinder can be read in either direction and because of the split made before constructing the strip, there is an additional point midway where the direction can be reversed without affecting the sound (Altschuler and Phillips 2015, 58).

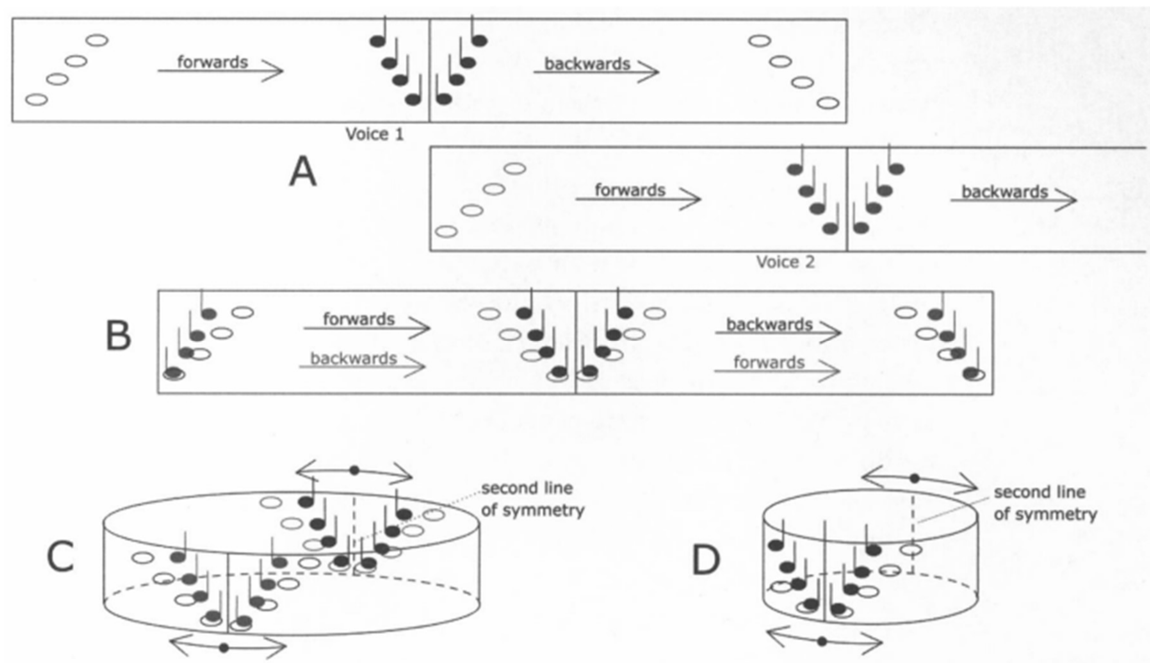


Figure 3.4.1. An analysis of Bach's Crab Canon provided by Altschuler and Phillips (2015, 59). A: The first voice plays the prime and then repeats itself backwards (performs the retrograde) while the second voice begins the prime at the start of voice one's retrograde. B: In the steady state, each voice plays the reverse of the other. C: With two different-sounding voices, the score topologically creates a cylinder while repeating. D: If both voices sound the same, the cylinder is wrapped around itself twice. In both cases of C and D, there is a second point where the piece can be played in either direction without changing the sound.

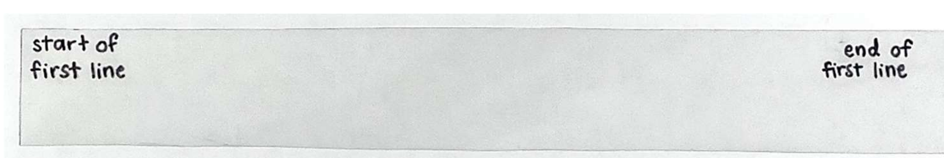


Figure 3.4.2. A strip containing the beginning and end of the first line.

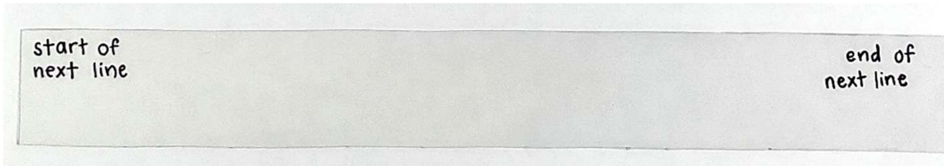


Figure 3.4.3. A strip containing the beginning and end of the second line.

Figure 3.4.4. A strip containing Figures 3.4.2 and 3.4.3 combined with the second line upside-down and facing away from the first line.

Figure 3.4.5. Any repeating piece of music can be read from a Möbius strip, so long as one line is on the front and the other is on the back, with each of the lines facing out. Once the ends of the strip seen in Figure 3.4.4 are connected by a twist, the end of the first line continues into the beginning of the next line and similarly, the end of the second line returns to the beginning of the first.

Knowing that any repeating piece of music can be simplified into a cylinder or double-wrapped cylinder (depending on whether the piece has distinct or indistinguishable voices respectfully), the argument that Bach's Crab Canon forms a Möbius strip starts to unravel.

v. A Contradiction Revealed

While the concept and YouTube video showing Bach's Crab Canon on a Möbius strip is provoking and fun to watch, the mathematical transformations utilized to create a Möbius strip and a perpetual crab canon are not the same.

Fundamentally, a Möbius strip is formed around the use of an inversion. It is thought of as one continuous strip, and while “direction” is relative once you have picked a starting point on the strip, one must continuously move in that singular direction for the Möbius strip to be achieved. Otherwise, it is almost equivalent to us unraveling the strip and undoing the twist, which defeats the purpose of creating a Möbius strip.

On the other hand, a crab canon is formed around the use of a retrograde. If focusing on a two-voice perpetual crab canon like Bach’s Crab Canon, both voices should cycle between playing the prime and its retrograde, with voice two creating the canonical form by beginning the prime when voice one reaches the beginning of the retrograde.

If we were to transpose Bach’s Crab Canon onto a Möbius strip without splicing it between the halfway point in the piece, reading the score “forwards” from the beginning of the prime would create a loop that cycles between the prime and its inversion, rather than the prime and its retrograde. As we have determined, a Möbius strip technically should only read in a single direction to maintain its continuous surface, however, in order to achieve the retrograde form needed in a crab canon, we would have to utilize reading both “forwards” and “backwards” on the strip. By doing so, if voice one reads forwards, this produces a cycle between the prime and its inversion, and if voice two reads backwards, this produces a cycle between the retrograde and the retrograde inversion. This output presents a different musical piece than the original score since we only desired parts that played the prime and the retrograde. The contradicting use of an inversion and a retrograde provokes an impossible task for a crab canon to be written on a continuous Möbius strip without splitting the score somewhere and creating a double-sided surface before twisting to emulate the infinite nature of the strip.

If we wanted to achieve the creation of a Möbius strip using some form of canon, both voices would need to read the piece in the same direction but begin at different times to create the canonical nature. Since a Möbius strip is centered around an inversion, a score would also need to have some relationship including an inversion so that a cylindrical piece could be placed about a Möbius strip. We know that a crab canon is unable to be placed on a Möbius strip due to its retrograde requirement, but other canons claim to work without needing to cut the score in half.

vi. Is There a Musical Möbius Strip?

In 1974, a collection of 14 canons (BWV 1087) was discovered within Bach's manuscript of the *Goldberg Variations* (BWV 988). Within these 14 canons, Canon 3 is composed using the eight notes of the Goldberg bass line, written as an inversion canon that Altschuler and Phillips argue can be placed on a Möbius strip (2015, 60). The score begins with voice one playing the prime and voice two entering with the inversion after voice one finishes the first four notes. Because of this relationship between prime and inversion, we can transpose the score onto a cylinder that wraps twice about a Möbius strip, alternating between voice one and two.

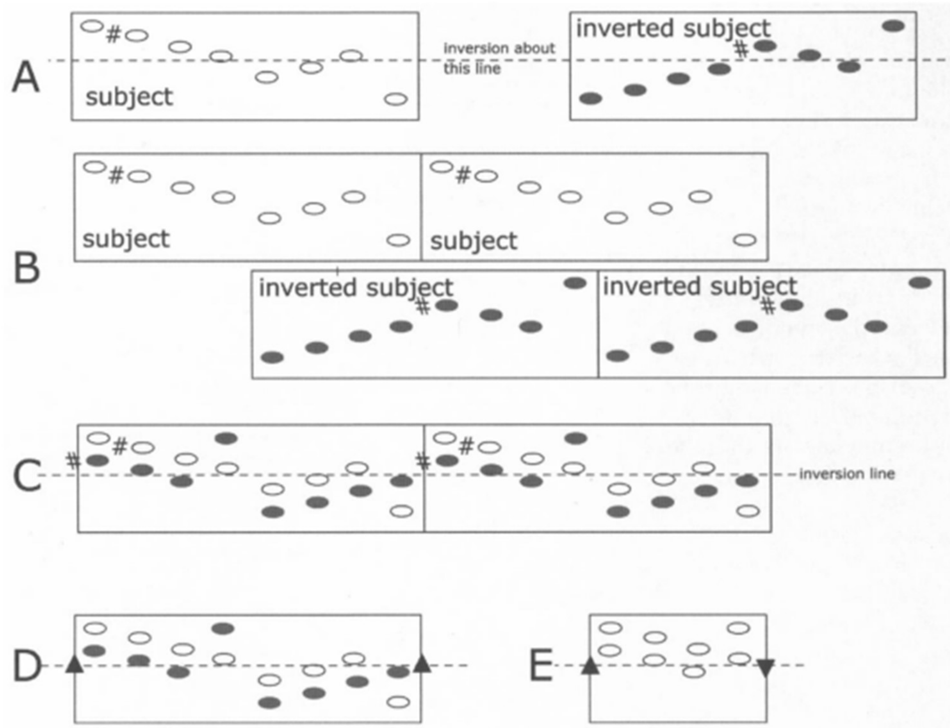


Figure 3.6.1. An analysis of Canon 3 from BWV 1087 provided by Altschuler and Phillips (2015, 60). A: The first voice plays the prime, which is the first eight notes of the Goldberg bass line. The prime is then inverted around the horizontal axis (the middle line of the staff). B: Voice two enters after the first four notes, playing the inversion. C: The steady state is reached after the first eight notes are played. D: With two distinct voices, the arrows indicate how the score forms a cylinder. E: If both voices sound the same, the cylinder in D can be wrapped around twice about a Möbius strip.

Figure 3.6.2. A transcribed version of Bach's original manuscript for Canon 3 from BWV 1087 with a segno indicating when the inversion begins.

Figure 3.6.3. The two voices split onto separate staves. Voice one (top) begins its first four notes alone. Voice two (bottom) is aligned so that it enters after voice one plays its fourth note.

Figure 3.6.4. Bach's Canon 3 on a Möbius strip.

When twisted into a Möbius strip, the canon can be read in a single direction and contains both voices without the need for splitting the strip. To enter the steady state as described in part C from Figure 3.6.1, the two voices must always be playing in opposition to one another with respect to the inversion. Thus, once voice two enters with the inversion after note four, it is clear to see that the canon can continuously read and cycle between the prime and its inversion about a Möbius strip.

vii. Musical Surfaces as a Visual Aid

Regardless of whether Bach's Crab Canon is truly a Möbius strip, translating musical writing onto a visual/geometric structure presents a unique way for us to understand the musical properties and choices that composers make (Cadeddu and Cannas 2017, 235). The Möbius strip is just one mathematical surface that we have identified as a tool to represent music. Topologically, there may be many more surfaces that can offer valuable visual representations of music. This visualization of music moving in time and space lends itself to analyzing how mathematical properties like transformations can be found in music. For further application of a musical Möbius strip and musical transformations, see section 5.

IV. Symmetry in Musical Composition

i. Introduction to Symmetry

Symmetry is not a new occurrence in musical compositions, but instead has been utilized by composers throughout the centuries. When we look at a musical plane compared to a two-dimensional Euclidean plane, the musical plane's restrictions result in less symmetry than the Euclidean plane (Hart 2009, 170). Specifically, if we think of a set of notes as a set of points in two-dimensional space, performing transformations on the set will be a little different than transformations on objects such as geometric shapes.

ii. Transformations in the Musical Plane

At its root, a transformation in mathematics simply means to change. The four main transformations that one becomes familiar with in geometry are rotation, translation, reflection, and dilation. However, these transformations behave differently in the musical plane due to a phrase's relationship to time and pitch. It is also important to note that while there are four main types of transformations, only rotations, translations, and reflections are considered isometries because they do not change the size or shape of a figure. Dilation is not an isometry since it creates a scaled version of the original figure.

These transformations must treat the two musical plane dimensions separately, as any horizontal change will affect time while any vertical change will affect pitch space. Thus, the primary forms of transformations used are translations and reflections. This results in the following possible isometries: horizontal translation (repetition), vertical translation (transposition), horizontal reflection (inversion about the horizontal axis), and vertical reflection (retrograde about the vertical axis) (Hart 2009, 170). Four additional combinations can be made from the four isometries. First, it is possible to perform a 180° rotation by combining both reflections to create a retrograde inversion. Likewise, combining both translations will allow for a motive to be repeated in a different key, which we call a transposition. Lastly, we can combine a translation with a reflection, which is known as a glide reflection in mathematical terms. Two different types of glide reflections can be produced: one reflects over the horizontal axis and then translates horizontally to create an inversion, while the other reflects over the vertical axis and translates vertically to create a retrograde in transposition (Hart 2009, 170).

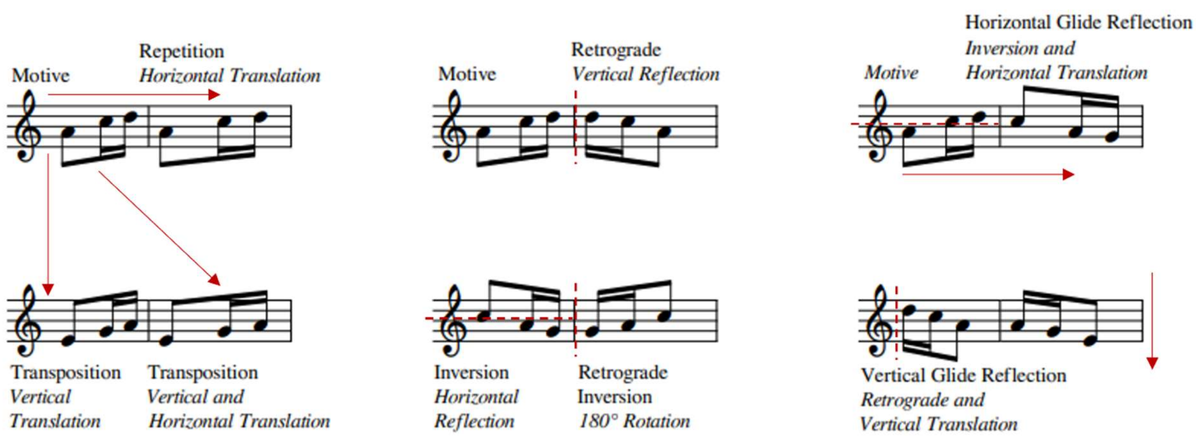


Figure 4.2.1. All eight of the isometric transformations in musical space. The arrows on the figure represent translations performed, while the dashed lines indicate the axes of reflection.

While the use of transformations seems intuitive, there is a level of ambiguity that comes with it. This ambiguity is similar to the uncertainty we account for in calculus when antidifferentiating. When integrating over an undefined interval, we add a constant to the resulting function in order to represent the family of functions whose derivatives are the initial function of integration. The uncertainty in musical space comes from how we define our axes of symmetry. Specifically, when we perform a reflection over the horizontal axis, we assume that the axis lies upon the middle line of the staff (e.g., this is where B5 lies within the treble clef). The choice of horizontal axis seems standardized, but consider how this changes as we change musical clefs. Does the horizontal axis stay on the middle staff line in bass or tenor clef, or does it move in relationship to where middle C (C4) is?

When sightreading a piece using the musical syllables of solfege, two different approaches can be used: one involves anchoring the syllable "do" to the note C, while the other uses a movable "do" that fixes itself based on the tonic of the key signature. In this way, is it also possible to move the horizontal axis to change the results of a musical reflection?

If we refer back to section 2.6, there is some clarity on the axes used to create inversions. Most often in Western tonal music, inversions occur with respect to the octave, ensuring that the sum of the semitones for the starting interval and the inverted interval total 12. Thus, unless we specify a different interval for inversions, it is common practice to invert about the middle line of the staff.

However, it may be that just as we can consider a coordinate system the anchor for a geometric figure in mathematics, we may think of a key signature as a similar anchor for a composition (Maor and Mathematics Teacher 1979, 419). It may be that once a key signature is established, musical transformations, such as inversions, can be performed in relation to the tonal center of the key. This idea of an unstandardized horizontal axis of inversion is more typical for compositions that date after the emancipation of tonality during the 1920s.

This idea of a moveable axis of reflection seems fair, as transpositions to a different key also rely on the initial key of a composition. And notice that just as a geometrical translation requires the initial shape or figure to remain the same, in transposition the intervals must also stay the same (Maor and Mathematics Teacher 1979, 419). Thus, so long as we establish this rule of continuity in the intervallic relationships, it could be that a movable horizontal axis can add to the way composers utilize musical transformations in their works.

iii. The Klein Four-Group in Music

In group theory, the Klein four-group is a group or set containing four elements in which we perform a binary operation to observe specific symmetries within the group. The Klein four-group is most often defined as $V = \{e, a, b, c\}$ with four axioms that can be observed:

1. Identity
   a. There is an identity element e such that for any non-identity element a of the group, $a * e = e * a = a$.
   b. Additionally, multiplying the identity element by itself will produce the identity $e * e = e$.
2. Associativity
   a. If a, b, and c are in the group, then $(a * b) * c = a * (b * c)$.
3. Inverses
   a. For any element a in the group, there is an element a^{-1} such that multiplying the element by its inverse will produce the identity element $a * a^{-1} = e$ and $a^{-1} * a = e$.
4. Closure
   a. If a and b are in the group, then the resulting element of $a * b$ is also in the group. Thus, all elements are contained within the group and can be produced by some combination of the other elements under the group operation.

Finally, the group operation is commutative, meaning that the order in which the terms are written does not affect the resulting element under the group operation. Because of this commutative property, we define the Klein four-group as abelian.

We can transpose this idea of the Klein four-group over to the musical transformations defined in the previous section. In coherence with the four axioms listed above, the musical Klein four-group can be identified as the group containing the prime, retrograde, inversion, and retrograde inversion (or $\{P, R, I, RI\}$). Here, the prime acts as our identity element, and each of the four axioms can be confirmed through a similar "musical" group operation.

	P	***R***	***I***	***RI***
P	P	R	I	RI
R	R	P	RI	I
I	I	RI	P	R
RI	RI	I	R	P

Table 4.3.1. A Cayley table showing the four elements prime (P), retrograde (R), inversion (I), and retrograde inversion (RI) as a closed Klein four-group.

iv. The Significance of Symmetry

Symmetry is a fundamental tool that we look for in nature to explain and understand the very facets of life. Our recognition of spatial and temporal relationships helps us to organize experiences, as

> We grasp varied experiences by viewing complex structures as combinations of simpler ones, reducing the amount of information to be processed. Symmetry

> allows us to apprehend objects and events as a synthesis of matching components, coordinating our field of perception and abetting our memory; above all, it invites us to see wholes as the necessary outcome of a joining of complementary parts (Morgan 1998, 1).

As spatial and temporal symmetries broaden, we eventually reach the development of musical forms. We can then distinguish musical time from musical space by recognizing how musical space—which encompasses both pitch and rhythm—fills the time consumed by musical events (Morgan 1998, 12). From these temporal and spatial components, their interactions can determine the degree of symmetry that is achieved in a composition. If both musical time and space are symmetrical, that is, repeating periodically, there is complete symmetry. Alhough, there can also be less formal symmetry where only one component is symmetrical. Since the repetition of a pitch-durational pattern always generates the time required to contain it, musical space (or content) symmetry always produces temporal symmetry. Temporal symmetry, however, can appear with either exact or only partial content duplication, thus this symmetry is not complete (Morgan 1998, 13). This interaction between musical space and time can create musical forms such as the canons we have been exploring so far, and many more forms utilize symmetries on a grander scale.

Not only does the use of symmetry create complexity in composition, but it also enhances our perception of aesthetics in music. Symmetry, like consonance, is pleasing to many Western-trained ears as there is a “satisfaction” in hearing clear uses of symmetries such as repetition or transposition. Therefore, it is not surprising that symmetrical concepts are applied to music due to their reliance on repetition, contrast, and other patterns.

Expanding our scope of musical analysis to a mathematical lens, mathematical surfaces create a new opportunity for audiences to achieve aural and visual satisfaction. Adding visually pleasing elements to already-pleasing auditory elements can augment the pleasure of hearing musical compositions. For many, our perception of the world around us relies on these aural and visual aesthetics to identify beauty in patterns. By combining the use of mathematical surfaces and musical composition, we open opportunities for audiences to become fully immersed—or at least appreciative—of the musical choices of composers.

V. Composition

i. The Whimsical Crab Cake Canon

Now with all of the knowledge we have collected on musical symmetries and transformations and their role in the creation of canons on a Möbius strip, it is time to compose our own piece.

For this composition, I first laid out my constraints and goals that needed to be met in order for the Möbius strip to function properly. To be a canon, I had to include at least two voices with some form of time or pitch variation. My piece also had to be invertible for it to transpose onto a Möbius strip. Lastly, though the piece could be a hodgepodge of notes, I wanted the composition to still contain a balance of consonance. I recognized that this piece was bound to contain some instances of dissonant intervals as it does not follow a chord progression or compositional form, but the goal was to create a melody that is relatively "aesthetically enjoyable" to Western-trained ears such as my own. Therefore, while it could be argued that this piece is tonally centered around the key of a minor, there is more of an ambiguity that plays into the creative nature of the composition. I also chose to invert all intervals with respect to their interval equivalents rather than their exact intervals in the diatonic scale for simplicity's sake.

"The Whimsical Crab Cake Canon" is a two-part canon in eight measures. Though the score of the canon is only eight measures, the full voicing is not heard until the end of the eighteenth measure, as measure nineteen allows for the canon to enter the steady state as a perpetual canon. Voice one enters alone in measure one by establishing the melody (the prime). Voice two enters in measure three to create the canon effect, with two of the five canon elements being used: (1) the time between the canon is two measures, and (2) the canon begins in a transformation of an inversion. When placed about a Möbius strip, the two voices can be played simultaneously while still being read in the same direction, but similarly to Bach's Canon 3, voice two will enter in inversion at the segno marked on beat one in measure three.

Figure 5.1.1. The Whimsical Crab Canon with a segno indicating when the inversion begins.

Figure 5.1.2. Voice one with a repeat written out. Measure 17 indicates one cycle of the prime and its inversion.

Figure 5.1.3. Voice two with a repeat written out. The canon enters at measure 3, two measures after voice one. Measure 19 indicates one cycle of the inversion and its prime.

Figure 5.1.4. Voices one and two written out separately in one system. After measure 18, the canon enters the steady state where each voice alternates between playing the prime and the inversion.

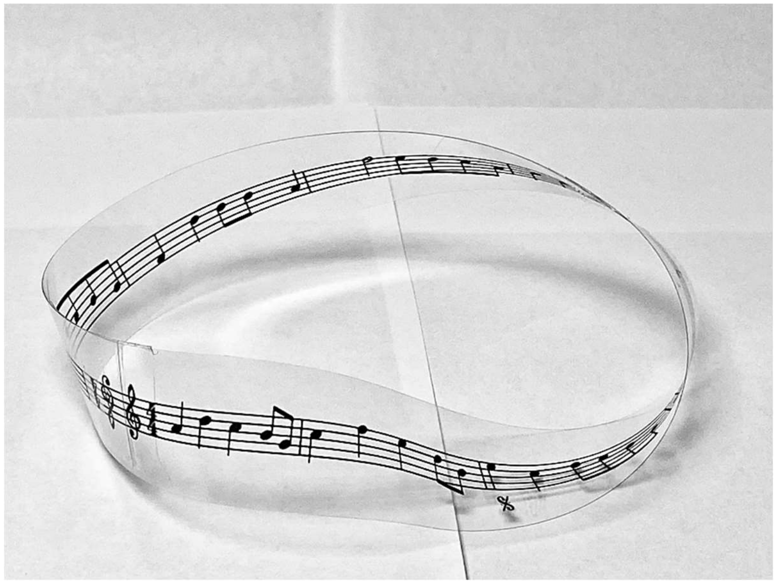


Figure 5.1.5. The Whimsical Crab Canon on a Möbius strip.

The above figures outline my compositional thought process as I developed The Whimsical Crab Canon. My goal for this composition was not to make the most poised, tonal piece of music, but to instead show the creativity and challenge of creating a canon that fits onto a Möbius strip. While I am certainly no Johann Sebastian Bach, I enjoyed the process of composing with a mathematical intention and focusing on elements that I often take for granted while listening to music. Had I had more time to sit down and compose a piece that could properly resolve on perfect cadences, maybe I would have—but then maybe we wouldn't have gotten the whimsical crab canon, and instead have gotten the sensible crab canon.

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